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A Minkowski Theorem for Quasicrystals

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Abstract The aim of this paper is to generalize Minkowski's theorem. This theorem is usually stated for a centrally symmetric convex body and a lattice both included in $\mathbf{R}^n$. In some situations, one may replace the lattice by a more general set for which a notion of density exists. In this paper, we prove a Minkowski theorem for quasicrystals, which bounds from below the frequency of differences appearing in the quasicrystal and belonging to a centrally symmetric convex body. The last part of the paper is devoted to quite natural applications of this theorem to Diophantine approximation and to discretization of linear maps.

Keywords Minkowski theorem · Quasicrystals · Diophantine approximation · Discretization

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1 Introduction

Minkowski theorem states that if a convex subset of $\mathbf{R}^n$ is centrally symmetric with respect to 0 and has a big enough volume, then it contains a non-trivial point with integer coordinates, i.e. a point of $\mathbf{Z}^n$. This result was proved by H. Minkowski in 1889, and initiated a whole field, now called *geometry of numbers* (see for example the books [4,12,15] or [2]). Since then, this theorem has led to many applications in various fields such as algebraic number theory, Diophantine approximation, harmonic analysis or complexity theory.

The goal of the present paper is to state a Minkowski theorem in the more general context where the lattice $\mathbf{Z}^n$ is replaced by a quasicrystal of $\mathbf{R}^n$. More precisely, we will be interested in sets satisfying the following definition.

Definition 1.1 Let $\Gamma \subset \mathbf{R}^n$ be a discrete set.

- The *upper density* of Γ is:

$$D^+(\Gamma) = \overline{\lim_{R\to+\infty}} \frac{\#(B_R \cap \Gamma)}{\mathrm{Vol}(B_R)},$$

  where B_R stands for the ball of radius R centred at 0.
- The *set of differences* of Γ is defined as

$$\Delta\Gamma = \Gamma - \Gamma = \{\gamma_1 - \gamma_2 \mid \gamma_1, \gamma_2 \in \Gamma\}.$$

- We will say that Γ is *Minkowski compatible* if $D^+(\Gamma) \in]0, +\infty[$ and $\Delta\Gamma$ is discrete.

Note that Minkowski compatible sets include Meyer sets (see [8]). Given a Minkowski compatible set $\Gamma \subset \mathbf{R}^n$ and a centrally symmetric convex body S, it is always possible to remove a finite subset from Γ such that the resulting set Γ' is still Minkowski compatible and satisfies $\Gamma' \cap S = \emptyset$. Therefore, one cannot hope to get a meaningful statement of Minkowski theorem involving only the number of points in $S \cap \Gamma$ and $D^+(\Gamma)$. The solution is to average upon the whole set, and to introduce the so-called *frequency of differences*. The frequency of the difference $u \in \mathbf{R}^n$ is defined as the density $\rho_\Gamma(u)$ of the set $\Gamma \cap (\Gamma - u)$ divided by the density of Γ (Definition 2.1). Again, the fact that the set Γ is Minkowski compatible is important here: it implies that the support of ρ_Γ is discrete. The main result of this paper is the following (Theorem 3.2).

Theorem 1.2 *Let $\Gamma \subset \mathbf{R}^n$ be a Minkowski compatible set, and $S \subset \mathbf{R}^n$ be a centrally symmetric convex body. Then*

$$\sum_{u\in S} \rho_\Gamma(u) \geq D^+(\Gamma)\mathrm{Vol}(S/2).$$

This theorem brings a new insight to the classical Minkowski theorem. It shows that the object of interest is in fact the set of differences of elements in Γ (which, for a lattice, is equal to Γ). In some sense, this point of view is already present in the original proof of Minkowski and the proof proposed in the sequel critically uses this fact. This is the purpose of Sect. 3.

In Sect. 4, we define *weakly almost periodic* sets (see Definition 4.1). We then state some nice properties of such sets in the view of the application of our main theorem, in particular the uniform upper density and the frequency of differences of a weakly almost periodic set are defined as limits (and no longer as upper limits). Roughly speaking, a set Γ is weakly almost periodic if given any ball B large enough, the intersection of Γ with any translate $t(B)$ of B is a translation of $B \cap \Gamma$ up to a proportion of points ε arbitrarily small. Such sets include a large class of quasicrystals, in particular model sets (and, of course, lattices).

The remaining part of the paper is dedicated to two applications of our main theorem. We investigate Diophantine approximation in Sect. 5.1. A corollary is derived to show the existence of a couple of points in a quasicrystal for which the slope of the line defined by those points is arbitrarily close to a fixed, chosen slope. Another application is also considered: for any given irrational number α and any positive number ε, the set E_α^ε of integers n such that $n\alpha$ is ε-close to 0 is a weakly almost periodic set. Hence, estimates of the mean number of points in E_α^ε which lie in the "neighbourhood" $[x-d, x+d]$ of a point $x \in E_\alpha^\varepsilon$ can be given.

In Sect. 5.2, a second application deals with discretizations of linear isometries. In particular, it shows that in most cases, it is impossible not to lose information while performing discrete rotations of numerical images with a naive algorithm.

2 Definitions

We begin with a few notations. The symmetric difference of two sets A and B will be denoted by $A \Delta B = (A \backslash B) \cup (B \backslash A)$.

We will use $\#A$ for the cardinality of a set A, λ for the Lebesgue measure on $\mathbf{R}^n$ and $\mathbf{1}$ for the indicator function. The number $\lceil x \rceil$ will denote the smallest integer bigger than x. For a set $A \subset \mathbf{R}^n$, we will denote by $\mathrm{Vol}(A)$ the volume of the set A. Recall that the notation B_R will refer to the ball of radius R centred at 0. Finally, for any integer n, the number μ_n will refer to the volume of the unit ball of dimension n. We will often use the notation $\sum_{x \in A} f(x)$ with A an uncountable set with no further justification: in this paper, every such f considered will be non-negative with countable support.

Definition 2.1 For every $v \in \mathbf{R}^n$, we set

$$\rho_\Gamma(v) = \frac{D^+\{x \in \Gamma \mid x + v \in \Gamma\}}{D^+(\Gamma)} = \frac{D^+(\Gamma \cap (\Gamma - v))}{D^+(\Gamma)} \in [0, 1]$$

the *frequency* of the difference v in the Minkowski compatible set Γ.

It is immediate to see that the support of ρ_Γ is discrete as it is included in the set $\Delta\Gamma$. Remark that when Γ is a lattice, the set $\Gamma \cap (\Gamma - v)$ is either equal to Γ (when $v \in \Gamma$), either empty (when $v \notin \Gamma$). Hence, $\rho_\Gamma(v) = \mathbf{1}_{v \in \Gamma}$ and for any subset A of $\mathbf{R}^n$, $\sum_{v \in A} \rho_\Gamma(v)$ counts the number of elements of Γ falling in A.

Definition 2.2 We say that the function f admits a *mean* $\mathcal{M}(f)$ if for every $\varepsilon > 0$, there exists $R_0 > 0$ such that for every $R \geq R_0$ and every $x \in \mathbf{R}^n$, we have (whenever the sum makes sense)

$$\left|\mathcal{M}(f) - \frac{1}{\mathrm{Vol}(B(x,R))} \sum_{v \in B(x,R)} f(v)\right| < \varepsilon.$$

3 A Minkowski Theorem for Quasicrystals

We now state a Minkowski theorem for the map ρ_Γ. To begin with, we recall the classical Minkowski theorem which is only valid for lattices (see for example IX.3 of [1] or the whole books [2,4,15]).

Theorem 3.1 (Minkowski) *Let* Λ *be a lattice of* $\mathbf{R}^n$, $k \in \mathbf{N}$ *and* $S \subset \mathbf{R}^n$ *be a centrally symmetric convex body. If* $\mathrm{Vol}(S/2) > k\,\mathrm{covol}(\Lambda)$*, then* S *contains at least* $2k$ *distinct points of* $\Lambda\backslash\{0\}$.

In particular, if $\mathrm{Vol}(S/2) > \mathrm{covol}(\Lambda)$, then S contains at least one point of $\Lambda\backslash\{0\}$. This theorem is optimal in the following sense: for every lattice Λ, there exists a centrally symmetric convex body S such that $\mathrm{Vol}(S/2) = k\,\mathrm{covol}(\Lambda)$ and that S contains less than $2k$ distinct points of $\Lambda\backslash\{0\}$.

Proof of Theorem 3.1 We consider the integer valued function

$$\varphi = \sum_{\lambda \in \Lambda} \mathbf{1}_{\lambda + S/2}.$$

The hypothesis about the covolume of Λ and the volume of $S/2$ imply that the mean[1] of the periodic function φ satisfies $\mathcal{M}(\varphi) > k$. In particular, as φ has integer values, there exists $x_0 \in \mathbf{R}^n$ such that $\varphi(x_0) \geq k+1$ (note that this argument is similar to pigeonhole principle). So there exists $\lambda_0, \dots, \lambda_k \in \Lambda$, with the λ_i sorted in lexicographical order (for a chosen basis), such that the $x_0 - \lambda_i$ all belong to $S/2$. As $S/2$ is centrally symmetric, as $\lambda_i - x_0$ belongs to $S/2$ and as $S/2$ is convex, $((x_0 - \lambda_0) + (\lambda_i - x_0))/2 = (\lambda_i - \lambda_0)/2$ also belongs to $S/2$. Then, $\lambda_i - \lambda_0 \in (\Lambda\backslash\{0\}) \cap S$ for every $i \in \{1, \dots, k\}$. By hypothesis, these k vectors are all different. To obtain $2k$ different points of $S \cap \Lambda\backslash\{0\}$ (instead of k different points), it suffices to consider also the points $\lambda_0 - \lambda_i$; this collection is disjoint from the collection of $\lambda_i - \lambda_0$ because the λ_i are sorted in lexicographical order. This proves the theorem. □

Minkowski theorem can be seen as a result about the function ρ_Γ. Recall that for a lattice Λ, $\sum_{u \in S} \rho_\Lambda(u)$ equals exactly the number of elements of $S \cap \Lambda$. Then, for a centrally symmetric convex body $S \subset \mathbf{R}^n$,

$$\sum_{u \in S} \rho_\Lambda(u) \geq 2\lceil D(\Lambda)\mathrm{Vol}(S/2)\rceil - 1.$$

[1] Here, the mean is taken with respect to Lebesgue measure instead of counting measure as in Definition 2.2.

Simply remark that the optimal k in Theorem 3.1 is given by $k = \lceil D(\Lambda)\mathrm{Vol}(S/2)\rceil - 1$. The following result is the main theorem of the paper.

Theorem 3.2 *Let Γ be a Minkowski compatible subset of $\mathbf{R}^n$, and $S \subset \mathbf{R}^n$ be a centrally symmetric convex body. Then*

$$\sum_{u\in S}\rho_\Gamma(u) \geq D^+(\Gamma)\mathrm{Vol}(S/2).$$

Remark 3.3 One can note that this theorem does not involve the factor 2 present in the classical Minkowski theorem which results of the fact that to any point of a lattice falling in the centrally symmetric set S corresponds its opposite, which also lies in S. The authors do not know if this factor 2 should or should not be present in Theorem 3.2 and this fact still remains to be investigated.

Proof of Theorem 3.2 The strategy of proof of this theorem is similar to that of the classical Minkowski theorem: we consider the set $\Gamma + S/2$ and define a suitable auxiliary function which depends on this set. The argument is based on a double counting for the quantity

$$\rho_a^R = \frac{1}{\mathrm{Vol}(B_R)}\sum_{v\in B_R\cap\Gamma}\mathbf{1}_{v\in(S/2+a)}\sum_{u\in S\cap\Delta\Gamma}\frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D^+(\Gamma)}. \tag{1}$$

Since the set Γ is discrete, the indexes in the two sums of (1) take a finite number of values. When R is large, ρ_a^R can be interpreted as the approximate frequency of the differences falling in S with the restriction that one of the points (in the difference) is in $S/2 + a$. It can also be interpreted as the local approximate frequency in a neighbourhood of a (the neighbourhood $S/2 + a$). The convenience of this restriction is expressed in (3). A way to get a global expression, from this local definition of the frequency, is to sum over $a \in \mathbf{R}^n$.

$$\begin{aligned}\int_{\mathbf{R}^n}\rho_a^R d\lambda(a) &= \frac{1}{\mathrm{Vol}(B_R)}\sum_{v\in B_R\cap\Gamma}\sum_{u\in S\cap\Delta\Gamma}\frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D^+(\Gamma)}\int_{\mathbf{R}^n}\mathbf{1}_{a\in(S/2+v)}d\lambda(a)\\ &= \frac{1}{\mathrm{Vol}(B_R)}\sum_{v\in B_R\cap\Gamma}\sum_{u\in S\cap\Delta\Gamma}\frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D^+(\Gamma)}\mathrm{Vol}(S/2)\\ &= \mathrm{Vol}(S/2)\sum_{u\in S\cap\Delta\Gamma}\frac{1}{\mathrm{Vol}(B_R)}\sum_{v\in B_R\cap\Gamma}\frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D^+(\Gamma)}.\end{aligned}$$

Thus, by the definition of ρ_Γ, we get

$$\overline{\lim_{R\to+\infty}}\int_{\mathbf{R}^n}\rho_a^R d\lambda(a) \leq \mathrm{Vol}(S/2)\sum_{u\in S\cap\Delta\Gamma}\rho_\Gamma(u). \tag{2}$$

In sight of the last inequality, it remains to show a lower bound on the left hand side. First of all, we remark that as S is a centrally symmetric convex body, $x, y \in S/2$

implies that $x + y \in S$, thus

$$\mathbf{1}_{v\in S/2+a}\mathbf{1}_{u\in S} \geq \mathbf{1}_{v\in S/2+a}\mathbf{1}_{u+v\in S/2+a}. \tag{3}$$

Hence, multiplying both sides by $\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}$, we get

$$\mathbf{1}_{v\in(S/2+a)\cap\Gamma}\mathbf{1}_{u\in S}\mathbf{1}_{u+v\in\Gamma} \geq \mathbf{1}_{v\in(S/2+a)\cap\Gamma}\mathbf{1}_{u+v\in(S/2+a)\cap\Gamma}.$$

We now sum this inequality over $u \in \Delta\Gamma$ to get

$$\sum_{u\in S\cap\Delta\Gamma}\mathbf{1}_{v\in(S/2+a)\cap\Gamma}\mathbf{1}_{u+v\in\Gamma} \geq \mathbf{1}_{v\in(S/2+a)\cap\Gamma}\sum_{u\in\Delta\Gamma}\mathbf{1}_{u+v\in(S/2+a)\cap\Gamma}.$$

Remarking that for every $v \in \Gamma$, every $v' \in \Gamma$ can be written as $v' = u + v$ with $u \in \Delta\Gamma$, we deduce that

$$\begin{aligned}\sum_{u\in S\cap\Delta\Gamma}\mathbf{1}_{v\in(S/2+a)\cap\Gamma}\mathbf{1}_{u+v\in\Gamma} &\geq \mathbf{1}_{v\in(S/2+a)\cap\Gamma}\sum_{v'\in\Gamma}\mathbf{1}_{v'\in(S/2+a)\cap\Gamma}\\ &= \mathbf{1}_{v\in(S/2+a)\cap\Gamma}\#((S/2+a)\cap\Gamma),\end{aligned}$$

and finally,

$$\rho_a^R \geq \frac{1}{D^+(\Gamma)}\frac{1}{\mathrm{Vol}(B_R)}\sum_{v\in B_R\cap\Gamma}\mathbf{1}_{v\in(S/2+a)}\#((S/2+a)\cap\Gamma).$$

We denote by B_R^S the S-interior of B_R and by $\overline{B}_R^S$ the S-expansion of B_R,

$$\begin{aligned}B_R^S &= (B_R^{\complement} + S)^{\complement} = \{x \in B_R \mid \forall s \in S, x + s \in B_R\},\\ \overline{B}_R^S &= B_R + S = \{x + s \mid x \in B_R, s \in S\}.\end{aligned}$$

In particular, $a \in B_R^S$ implies that $S/2 + a \subset B_R$ and $a \in B_R$ implies that $S/2 + a \in \overline{B}_R^S$. Then

$$\begin{aligned}\int_{\mathbf{R}^n}\rho_a^R d\lambda(a) &\geq \frac{1}{D^+(\Gamma)}\frac{1}{\mathrm{Vol}(B_R)}\int_{\mathbf{R}^n}\Big(\sum_{v\in B_R\cap\Gamma}\mathbf{1}_{v\in(S/2+a)}\#((S/2+a)\cap\Gamma)\Big)d\lambda(a)\\ &\geq \frac{1}{D^+(\Gamma)}\frac{1}{\mathrm{Vol}(B_R)}\int_{B_R^S}\Big(\sum_{v\in B_R\cap\Gamma}\mathbf{1}_{v\in(S/2+a)}\#((S/2+a)\cap\Gamma)\Big)d\lambda(a)\\ &\geq \frac{1}{D^+(\Gamma)}\frac{1}{\mathrm{Vol}(B_R)}\int_{B_R^S}\#((S/2+a)\cap\Gamma)^2 d\lambda(a).\end{aligned}$$

Using the convexity of $x \mapsto x^2$, we deduce that

$$\begin{aligned} &\overline{\lim_{R\to+\infty}} \int_{\mathbf{R}^n} \rho_a^R d\lambda(a) \\ &\quad \geq \overline{\lim_{R\to+\infty}} \frac{\operatorname{Vol}(B_R^S)}{D^+(\Gamma)\operatorname{Vol}(B_R)} \left(\frac{1}{\operatorname{Vol}(B_R^S)} \int_{B_R^S} \#((S/2+a)\cap\Gamma) d\lambda(a) \right)^2. \end{aligned} \tag{4}$$

We then use the fact that the family $\{B_R\}_{R>0}$ is van Hove when R goes to infinity (see for example [14, Eq. 4]), that is

$$\lim_{R\to+\infty} \frac{\operatorname{Vol}(B_R)-\operatorname{Vol}(B_R^S)}{\operatorname{Vol}(B_R)} = 0 \quad \text{and} \quad \lim_{R\to+\infty} \frac{\operatorname{Vol}(B_R)-\operatorname{Vol}(\overline{B}_R^S)}{\operatorname{Vol}(B_R)} = 0.$$

It remains to compute the remaining term in (4),

$$\frac{1}{\operatorname{Vol}(B_R^S)} \int_{B_R^S} \#((S/2+a)\cap\Gamma) d\lambda(a).$$

The quantity $\#((S/2+a)\cap\Gamma)$ is bounded by some constant M (as S can be included in some ball of large radius), independently from a and is equal to

$$\sum_{v\in B_R\cap\Gamma} \mathbf{1}_{v\in(S/2+a)\cap\Gamma} \quad \text{for all } a\in B_R^S.$$

Hence,

$$\begin{aligned} \frac{1}{\operatorname{Vol}(B_R^S)} \left| \int_{B_R^S} \#((S/2+a)\cap\Gamma) d\lambda(a) - \int_{\overline{B}_R^S} \sum_{v\in B_R\cap\Gamma} \mathbf{1}_{v\in S/2+a} d\lambda(a) \right| \\ \leq M \frac{\operatorname{Vol}(\overline{B}_R^S \setminus B_R^S)}{\operatorname{Vol}(B_R^S)}; \end{aligned}$$

thus the two integrals have the same limit superior when R tends to $+\infty$. Besides,

$$\begin{aligned} \frac{1}{\operatorname{Vol}(B_R^S)} \int_{\overline{B}_R^S} \sum_{v\in B_R\cap\Gamma} \mathbf{1}_{v\in S/2+a} d\lambda(a) &= \frac{1}{\operatorname{Vol}(B_R^S)} \sum_{v\in B_R\cap\Gamma} \int_{\overline{B}_R^S} \mathbf{1}_{a\in S/2+v} d\lambda(a) \\ &= \frac{1}{\operatorname{Vol}(B_R^S)} \sum_{v\in B_R\cap\Gamma} \operatorname{Vol}(S/2) \\ &= \frac{\operatorname{Vol}(B_R)}{\operatorname{Vol}(B_R^S)} \frac{\#(B_R\cap\Gamma)}{\operatorname{Vol}(B_R)} \operatorname{Vol}(S/2). \end{aligned}$$

Applied to (4), this gives

$$\overline{\lim_{R\to+\infty}} \int_{\mathbf{R}^n} \rho_a^R d\lambda(a) \geq \operatorname{Vol}(S/2)^2 D^+(\Gamma).$$

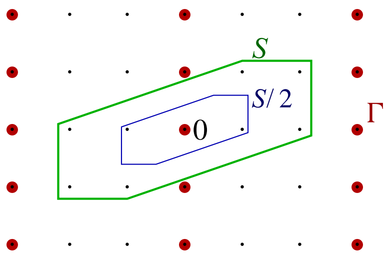

Fig. 1 Example 3.5 of equality case in Corollary 3.4 for $k = 3$

To finish the proof, we combine the last inequality with the first estimate of (2) and get

$$\sum_{u\in S\cap\Delta\Gamma} \rho_\Gamma(u) \geq \mathrm{Vol}(S/2)D^+(\Gamma). \qquad \square$$

We now give an alternative version of Theorem 3.2, where the volume of the set $S/2$ is replaced by the number of integer points it contains (a similar statement had already been obtained by M. Henk for sublattices of $\mathbf{Z}^n$ in [7]).

Corollary 3.4 *If $\Gamma \subset \mathbf{Z}^n$ is Minkowski compatible, then*

$$\sum_{u\in S} \rho_\Gamma(u) \geq D(\Gamma)\#(S/2 \cap \mathbf{Z}^n).$$

The idea of the proof of this corollary is identical to that of Theorem 3.2, but instead of integrating ρ_a^R (see (1)) over $\mathbf{R}^n$, one sums ρ_a^R over $\mathbf{Z}^n$.

The case of equality in this corollary is attained even in the non-trivial case where $\#(S/2 \cap \mathbf{Z}^n) > 1$, as shown by the following example.

Example 3.5 If k is an odd number, if Γ is the lattice $k\mathbf{Z} \times \mathbf{Z}$, and if S is a centrally symmetric convex set such that (see Fig. 1)

$$S \cap \mathbf{Z}^2 = \big\{(i, 0) \mid i \in \{-(k-1), \ldots, k-1\}\big\} \cup \big\{ \pm (i, 1) \mid i \in \{1, \ldots, k-1\}\big\},$$

then $\sum_{u\in S} \rho(u) = 1$, $D(\Gamma) = 1/k$ and $\#(S/2 \cap \mathbf{Z}^2) = k$.

4 Weakly Almost Periodic Sets

In this section, we assume that every set Γ considered is Minkowski compatible. We describe a family of discrete sets called *weakly almost periodic sets*. For these sets, the superior limits appearing in the definitions of the upper density and the frequency of differences are actually limits. Roughly speaking, a weakly almost periodic set Γ is a set for which two large patches are almost identical, up to a set of upper density smaller than ε. More precisely, we have the following definition.

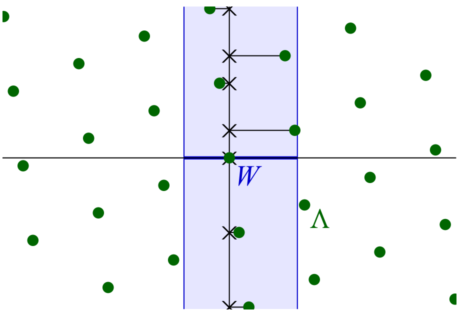

Fig. 2 Construction of a model set

Definition 4.1 We say that a set Γ is *weakly almost periodic* if for every $\varepsilon > 0$, there exists $R > 0$ such that for every $x, y \in \mathbf{R}^n$, there exists $v \in \mathbf{R}^n$ such that

$$\frac{\#((B(x, R) \cap \Gamma)\Delta((B(y, R) \cap \Gamma) - v))}{\mathrm{Vol}(B_R)} \leq \varepsilon. \tag{5}$$

Note that the vector v is different from $y - x$ *a priori*. Of course, every lattice, or every finite union of translates of a given lattice, is weakly almost periodic.

An important class of examples of weakly almost periodic sets is given by *model sets* (sometimes also called "cut-and-project" sets). These sets have numerous applications to theory of quasicrystals, harmonic analysis, number theory, discrete dynamics etc. (see for instance [10] or [13]).

Definition 4.2 Let Λ be a lattice of $\mathbf{R}^{m+n}$, p_1 and p_2 the projections of $\mathbf{R}^{m+n}$ on respectively $\mathbf{R}^m \times \{0\}_{\mathbf{R}^n}$ and $\{0\}_{\mathbf{R}^m} \times \mathbf{R}^n$, and W a Riemann integrable subset of $\mathbf{R}^m$. The *model set* modelled on the lattice Λ and the *window* W is (see Fig. 2)

$$\Gamma = \{p_2(\lambda) \mid \lambda \in \Lambda,\ p_1(\lambda) \in W\}.$$

In [5] it is proved that these sets are weakly almost periodic sets. Moreover, if the projection p_2 is injective when restricted to Λ, and the set $p_2(\Lambda)$ is dense, then the density of the obtained model set is equal to $\mathrm{Vol}(W)\,\mathrm{covol}(\Lambda)$ (see for example Proposition 4.4 of [11]).

A weakly almost periodic set possesses a uniform density, as stated by the following proposition of [5].

Proposition 4.3 *Let Γ be a weakly almost periodic set. Then there exists a number $D(\Gamma)$, called the* uniform density *of Γ, satisfying: for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for every $R > R_\varepsilon$ and every $x \in \mathbf{R}^n$,*

$$\left|\frac{\#(B(x, R) \cap \Gamma)}{\mathrm{Vol}(B(x, R))} - D(\Gamma)\right| < \varepsilon.$$

In particular, $D(\Gamma) = D^+(\Gamma)$, and for every $x \in \mathbf{R}^n$, we have

$$D(\Gamma) = \lim_{R \to +\infty} \frac{\#(B(x, R) \cap \Gamma)}{\mathrm{Vol}(B(x, R))}.$$

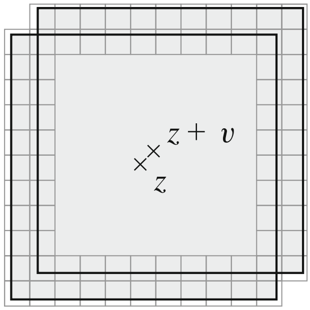

Fig. 3 Covering the set $B(z,R)\Delta B(z+v,R)$ by cubes of radius R_0 (the same process can be done for euclidean balls)

As noted in [5], it seems that the notion of weakly almost periodicity is the weakest that allows this uniform convergence of density.

The following lemma states that the occurrences of a given difference in a weakly almost periodic set form a weakly almost periodic set.

Lemma 4.4 *Let $v\in\mathbf{R}^n$ and Γ be a discrete weakly almost periodic set. Then the set*

$$\{x\in\Gamma \mid x+v\in\Gamma\}=\Gamma\cap(\Gamma-v)$$

is weakly almost periodic.

Proof Let $\varepsilon>0$ and $v\in\mathbf{R}^n$. As Γ is a weakly almost periodic set, for every $\varepsilon>0$, there exists $R>0$ such that for every $x,y\in\mathbf{R}^n$, there exists $w\in\mathbf{R}^n$ such that

$$\frac{\#((B(x,R)\cap\Gamma)\Delta((B(y,R)\cap\Gamma)-w))}{\operatorname{Vol}(B_R)}\leq\varepsilon. \tag{6}$$

On the other hand, taking $\varepsilon=1$ in Definition 4.1, we deduce that there exists $R_0>0$ such that for every $y\in\mathbf{R}^n$, we have

$$\#(B(y,R_0)\cap\Gamma)\leq\#(B_{R_0}\cap\Gamma)+\operatorname{Vol}(B_{R_0}).$$

By covering the symmetric difference $B(z,R)\Delta B(z+v,R)$ by balls of radius R_0 as in Fig. 3, we deduce that there exists a constant $C>0$ such that, for every $z\in\mathbf{R}^n$ (and in particular for x and y):

$$\begin{aligned}\frac{\#((B(z,R)\cap\Gamma)\Delta((B(z+v,R)\cap\Gamma)))}{\operatorname{Vol}(B_R)}&=\frac{\#((B(z,R)\Delta B(z+v,R))\cap\Gamma)}{\operatorname{Vol}(B_R)}\\&\leq C\frac{\operatorname{Vol}((B(z,R)\Delta B(z+v,R)))}{\operatorname{Vol}(B_R)}.\end{aligned}$$

Thus, considering a smaller ε if necessary, one can choose R arbitrarily large compared to $\|v\|$. In this case, we get,

$$\frac{\#((B(z,R)\cap\Gamma)\Delta((B(z+v,R)\cap\Gamma)))}{\operatorname{Vol}(B_R)}\leq\varepsilon. \tag{7}$$

Let $x, y \in \mathbf{R}^n$. We can now estimate the quantity

$$A \doteq \frac{\#((B(x,R)\cap\Gamma\cap(\Gamma-v))\Delta\big((B(y,R)\cap\Gamma\cap(\Gamma-v))-w))}{\mathrm{Vol}(B_R)}:$$

$$A \leq \frac{\#((B(x,R)\cap\Gamma)\Delta((B(y,R)\cap\Gamma)-w))}{\mathrm{Vol}(B_R)} + \frac{\#((B(x,R)\cap(\Gamma-v))\Delta((B(y,R)\cap(\Gamma-v))-w))}{\mathrm{Vol}(B_R)}.$$

The first term is smaller than ε by (6); and the second term (denoted by A_2) is equal to (by a translation of vector v)

$$A_2 = \frac{\#((B(x+v,R)\cap\Gamma)\Delta((B(y+v,R)\cap\Gamma)-w))}{\mathrm{Vol}(B_R)},$$

which by (6) and (7) leads to

$$A_2 \leq 3\varepsilon.$$

Finally, $A \leq 4\varepsilon$. □

When Γ is weakly almost periodic, we deduce, from Lemma 4.4 together with Proposition 4.3, that the upper limits appearing in the uniform upper density $D^+(\Gamma)$ and the frequencies of differences $\rho_\Gamma(v)$ are, in fact, limits. Moreover, ρ_Γ possesses a mean (see Definition 2.2) that can be computed easily.

Proposition 4.5 *If the conclusion of Proposition 4.3 holds, then*

$$\mathcal{M}(\rho_\Gamma) = D(\Gamma).$$

In particular, this is the case when the set Γ is Minkowski compatible and weakly almost periodic.

Proof This proof lies primarily in an inversion of limits.

Let $\varepsilon > 0$. As Γ satisfies the conclusion of Proposition 4.3, there exists $R_0 > 0$ such that for every $R \geq R_0$ and every $x \in \mathbf{R}^n$, we have

$$\left| D(\Gamma) - \frac{\Gamma \cap B(x,R)}{\mathrm{Vol}(B_R)} \right| \leq \varepsilon. \tag{8}$$

So, we choose $R \geq R_0$, $x \in \mathbf{Z}^n$ and compute

$$\frac{1}{\mathrm{Vol}(B_R)} \sum_{v\in B(x,R)} \rho_\Gamma(v) = \frac{1}{\mathrm{Vol}(B_R)} \sum_{v\in B(x,R)} \frac{D\big((\Gamma-v)\cap\Gamma\big)}{D(\Gamma)}$$

$$\begin{aligned}
&= \frac{1}{\mathrm{Vol}(B_R)} \sum_{v\in B(x,R)} \lim_{R'\to+\infty} \frac{1}{\mathrm{Vol}(B_{R'})} \sum_{y\in B_{R'}} \frac{\mathbf{1}_{y\in\Gamma-v}\mathbf{1}_{y\in\Gamma}}{D(\Gamma)}\\
&= \frac{1}{D(\Gamma)} \lim_{R'\to+\infty} \frac{1}{\mathrm{Vol}(B_{R'})} \sum_{y\in B_{R'}} \mathbf{1}_{y\in\Gamma} \frac{1}{\mathrm{Vol}(B_R)} \sum_{v\in B(x,R)} \mathbf{1}_{y\in\Gamma-v}\\
&= \frac{1}{D(\Gamma)} \underbrace{\lim_{R'\to+\infty} \frac{1}{\mathrm{Vol}(B_{R'})} \sum_{y\in B_{R'}} \mathbf{1}_{y\in\Gamma}}_{\text{first term}} \underbrace{\frac{1}{\mathrm{Vol}(B_R)} \sum_{v'\in B(y+x,R)} \mathbf{1}_{v'\in\Gamma}}_{\text{second term}} .
\end{aligned}$$

By (8), the second term is ε-close to $D(\Gamma)$. Considered independently, the first term is equal to $D(\Gamma)$ [still by (8)]. Thus, we have

$$\left|\frac{1}{\mathrm{Vol}\big(B(x,R)\big)} \sum_{v\in B(x,R)} \rho_\Gamma(v) - D(\Gamma)\right| \le \varepsilon,$$

which conclude the proof. □

5 Applications

5.1 Application to Diophantine Approximation

5.1.1 A Dirichlet Theorem for Quasicrystals

In this section, we develop a generalization of Dirichlet theorem for approximations of irrational numbers. We give this theorem for completeness.

Theorem 5.1 (Dirichlet) *Let $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be such that at least one of the α_i is irrational. Then there are infinitely many tuples of integers $(x_1, \dots, x_n, y)$ such that the highest common factor of $x_1, \dots, x_n, y$ is 1 and that*

$$\left|\alpha_i - \frac{x_i}{y}\right| \le y^{-1-1/n} \quad \textit{for } i = 1, \dots, n.$$

One may be interested by approximations of real numbers by tuples in sets different from $\mathbf{Z}^{n+1}$, for instance quasicrystals. The following result is an easy consequence of Theorem 3.2 which is convenient for the study of Diophantine approximations in weakly almost periodic sets.

Corollary 5.2 *Let $L_1, \dots, L_n$ be n linear forms on $\mathbf{R}^n$ such that* $\det(L_1, \dots, L_n) \neq 0$. *Let $A_1, \dots, A_n$ be positive real numbers and let Γ be a Minkowski compatible set. Then*

$$\sum_{x\,:\,\forall i,\ |L_i(x)|\le A_i} \rho_\Gamma(x) \ge D(\Gamma) A_1 \dots A_n |\det(L_1, \dots, L_n)|^{-1}.$$

Proof Apply Theorem 3.2 to $S = \{x \in \mathbf{R}^n \mid \forall i,\ |L_i(x)| \le A_i\}$. □

The following corollary is a version of Dirichlet theorem for weakly almost periodic sets.

Corollary 5.3 *Let $Q > 0$. Let $\Gamma \subset \mathbf{R}^{n+1}$ be a Minkowski compatible set and let $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be real numbers. Then there exist at least two points $v = (x_1^v, \dots, x_n^v, y^v)$ and $w = (x_1^w, \dots, x_n^w, y^w)$ in Γ such that*

$$\left|\alpha_i - \frac{x_i^v - x_i^w}{y^v - y^w}\right| \le \left(\frac{2}{D(\Gamma)}\right)^{1/n} |y^v - y^w|^{-1-1/n} \tag{9}$$

with the additional property that $|y^v - y^w| \le 2Q/D(\Gamma)$.

Proof Define for any $i \in \{1, \dots, n\}$ the linear form $L_i : \mathbf{R}^{n+1} \to \mathbf{R}$ by

$$L_i(x_1, \dots, x_n, y) = x_i - \alpha_i y$$

and $L_{n+1}(x_1, \dots, x_n, y) = y$. We have immediately that $\det(L_1, \dots, L_n) = 1$. We apply Corollary 5.2 with $A_1 = \dots = A_n = Q^{-1/n}$ and $A_{n+1} = 2Q/D(\Gamma)$, thus

$$\sum_{\substack{(x_1,\dots,x_n,y)\in\Delta\Gamma\backslash\{0\} \\ \forall i,\ |L_i(x)|\le A_i}} \rho_\Gamma(x) \ge 1.$$

In particular, there exists a point u in $\Delta\Gamma$ which is a difference of two different points $v = (x_1^v, \dots, x_n^v, y^v)$ and $w = (x_1^w, \dots, x_n^w, y^w)$ in the weakly almost periodic set Γ and such that $\forall i$, $|L_i(v - w)| \le A_i$. Then for each $i \le n$, $|L_i(v - w)| \le A_i$ and $|L_{n+1}(v - w)| \le A_{n+1}$ imply

$$\left|\alpha_i - \frac{x_i^v - x_i^w}{y^v - y^w}\right| \le \left(\frac{2}{D(\Gamma)}\right)^{1/n} |y^v - y^w|^{-1-1/n}$$

and $|y^v - y^w| \le 2Q/D(\Gamma)$. □

Let us state a direct consequence of Corollary 5.3 for $\Gamma \subset \mathbf{Z}^{n+1}$ and $\overline{\alpha} \notin \mathbf{Q}^n$. In this case, choose an index i_0 such that $\alpha_{i_0} \notin \mathbf{Q}$, and consider the line of $\mathbf{R}^2$ defined by $x_{i_0} - \alpha_{i_0} y = 0$. This line does not meet $\mathbf{Z}^2\backslash\{0\}$, so the smallest norm of the points $(x_{i_0}, y) \in \mathbf{Z}^2\backslash\{0\}$ satisfying $|x_{i_0} - \alpha_{i_0} y| \le Q^{-1/n}$ goes to infinity when Q goes to infinity. Thus, the smallest norm of the points $x \in \mathbf{Z}^{n+1}\backslash\{0\}$ satisfying $|L_i(x)| \le A_i$ for all i goes to infinity when Q goes to infinity (recall that $A_1 = \dots = A_n = Q^{-1/n}$ and $A_{n+1} = 2Q/D(\Gamma)$). By choosing a sequence $(Q_p)_p$ of positive numbers tending to infinity, we deduce that there exists an infinite number of couples (x_i^v, y^v) and (x_i^w, y^w) of points of Γ satisfying (9).

Remark that the approximation quality highly depends on the density of the considered set Γ. In particular, we will find at least one direction in Γ close to $\overline{\alpha}$ (close with a factor comparable to $Q^{-1-1/n}$) in one ball of size comparable with Q. This can be seen as a non-asymptotic counterpart of the strong results of [9] (Fig. 4).

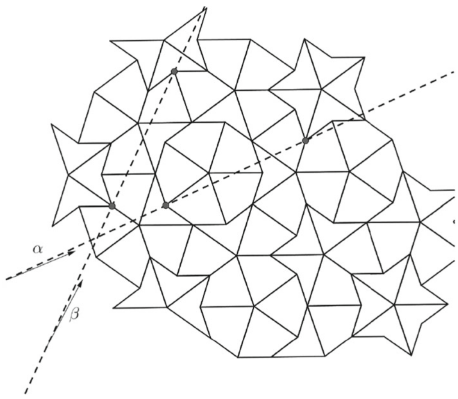

Fig. 4 For a fixed chosen direction α, one can find two points in a Penrose tilling defining a line whose slope is close to α. Two different chosen directions are shown. Penrose tilings are model sets, thus are weakly almost periodic (see [3])

5.1.2 Frequency of Differences and Approximations

Theorem 3.2 gives informations about the simultaneous approximations of a set of numbers for an arbitrary norm: given a norm N on $\mathbf{R}^n$ and a n-tuple of $\mathbf{Q}$-linearly independent numbers $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, we look at the set

$$\begin{aligned} E^{\varepsilon}_{\overline{\alpha}} &= \{y \in \mathbf{Z} \mid \exists x \in \mathbf{Z}^n : N(y\overline{\alpha} - x) < \varepsilon\} \\ &= \left\{y \in \mathbf{Z} \mid \exists x \in \mathbf{Z}^n : N\Big(\overline{\alpha} - \frac{x}{y}\Big) < \frac{\varepsilon}{y}\right\}. \end{aligned}$$

This set is a model set modelled on the lattice spanned by the matrix

$$\begin{pmatrix} -1 & & & \alpha_1 \\ & \ddots & & \vdots \\ & & -1 & \alpha_n \\ & & & 1 \end{pmatrix}$$

and on the window $W = \{x \in \mathbf{R}^n \mid N(x) < \varepsilon\}$, in particular it is a weakly almost periodic set. Its density can be easily computed: $E^{\varepsilon}_{\overline{\alpha}}$ is the set of $y \in \mathbf{Z}$ such that the projection of $y\overline{\alpha}$ on $\mathbf{R}^n/\mathbf{Z}^n$ belongs to $\mathrm{pr}_{\mathbf{R}^n/\mathbf{Z}^n}(W)$. But the rotation of angle $\overline{\alpha}$ on $\mathbf{R}^n/\mathbf{Z}^n$ is ergodic (as it forms a $\mathbf{Q}$-free family), so the density of $E^{\varepsilon}_{\overline{\alpha}}$ is equal to the area of $\mathrm{pr}_{\mathbf{R}^n/\mathbf{Z}^n}(W)$, which is equal to $\mathrm{Vol}(W)$ as long as W does not intersect any integer translate of itself. Then Theorem 3.2 asserts that for every $d > 0$,

$$\sum_{\substack{u \in \mathbf{Z} \\ |u| \le d}} \rho_{E^{\varepsilon}_{\overline{\alpha}}}(u) \ge d\,\mathrm{Vol}(W).$$

In other words, given $v \in E^{\varepsilon}_{\overline{\alpha}}$, the average number of points $v' \in E^{\varepsilon}_{\overline{\alpha}}$ such that $|v - v'| \le d$ is bigger than $d\,\mathrm{Vol}(W)$.

5.2 Application to the Dynamics of the Discretizations of Linear Maps

Here, we recall a theorem of [6] and sketch its proof, which crucially uses Minkowski theorem for weakly almost periodic sets.

We take a Euclidean projection[2] π of $\mathbf{R}^n$ onto $\mathbf{Z}^n$; given $A \in GL_n(\mathbf{R})$, the *discretization* of A is the map $\widehat{A} = \pi \circ A : \mathbf{Z}^n \to \mathbf{Z}^n$. This is maybe the simplest way to define a discrete analogue of a linear map. We want to study the action of such discretizations on the set $\mathbf{Z}^n$; in particular if these maps are far from being injective, then when applied to numerical images, discretizations will induce a loss of quality in the resulting images.

Thus, we study the *rate of injectivity* of discretizations of linear maps: given a sequence $(A_k)_{k\in\mathbf{N}}$ of linear maps, the *rate of injectivity in time* k of this sequence is the quantity

$$\tau^k(A_1, \ldots, A_k) = \lim_{R\to+\infty} \frac{\#((\widehat{A_k} \circ \ldots \circ \widehat{A_1})(B_R \cap \mathbf{Z}^n))}{\#(B_R \cap \mathbf{Z}^n)} \in]0, 1].$$

To prove that the limit of this definition is well defined, we show that

$$\limsup_{R\to+\infty} \frac{\#((\widehat{A_k} \circ \ldots \circ \widehat{A_1})(B_R \cap \mathbf{Z}^n))}{\#(B_R \cap \mathbf{Z}^n)} = |\det(A_1 \ldots A_k)| D^+((\widehat{A_k} \circ \ldots \circ \widehat{A_1})(\mathbf{Z}^n))$$

and use the fact that the set $(\widehat{A_k} \circ \ldots \circ \widehat{A_1})(\mathbf{Z}^n)$ is weakly almost periodic. In particular, when all the matrices are of determinant ± 1, we have

$$\tau^k(A_1, \ldots, A_k) = D^+((\widehat{A_k} \circ \ldots \circ \widehat{A_1})(\mathbf{Z}^n))$$

Then, Theorem 3.2 applies to prove the next result.

Theorem 5.4 *Let $(P_k)_{k\geq 1}$ be a generic[3] sequence of matrices[4] of $O_n(\mathbf{R})$. Then*

$$\tau^k((P_k)_{k\geq 1}) \underset{k\to+\infty}{\longrightarrow} 0.$$

Thus, for a generic sequence of angles, the application of successive discretizations of rotations of these angles to a numerical image will induce an arbitrarily large loss of quality of this image (see Fig. 5).

Let us explain why the proof of Theorem 5.4 (which can be found as a whole in [6]) requires Theorem 3.2. The idea is to study the set of differences of the sets

$$\Gamma_k = (\widehat{P_k} \circ \ldots \circ \widehat{P_1})(\mathbf{Z}^n).$$

2 That is, $\pi(x)$ is (one of the) point(s) of $\mathbf{Z}^n$ the closest from x for the Euclidean norm.

3 A property concerning elements of a topological set X is called *generic* if it is satisfied on at least a countable intersection of open and dense sets. In particular, Baire theorem implies that if this space is complete (as here), then this property is true on a dense subset of X.

4 The set of sequences of matrices is endowed with the norm $\|(P_k)_{k\geq 1}\| = \sup_{k\geq 1} \|P_k\|$, making it a complete space (see Note 3).

Fig. 5 Original image (*left*) of size 220 × 282 and ten successive random rotations of this image (*right*), obtained with the software *Gimp* (linear interpolation algorithm)

The first step is to prove that these sets are weakly almost periodic, to be able to prove some uniform distribution results ([6, Thm. 2 and Lem. 4]). Then, by analysing the action of the discretization of a generic map on the frequency of differences, one can prove the following lemma, obtained by combining Proposition 4 (ii)[5] and the second paragraph of [6, p. 83].

Lemma 5.5 *For every k, for every isometry $P \in O_n(\mathbf{R})$ and every $\varepsilon > 0$, there exist $\delta > 0$ and a matrix $Q \in O_n(\mathbf{R})$ such that $d(P, Q) < \varepsilon$ satisfying: for every $v_0 \in \mathbf{Z}^n$,*

(i) *either there exists $v_1 \in \mathbf{Z}^n \backslash \{0\}$ such that $\|v_1\|_2 < \|v_0\|_2$ and that*

$$\rho_{\widehat{Q}(\Gamma_k)}(v_1) \geq \delta \rho_{\Gamma_k}(v_0);$$

(ii) *or*

$$D(\widehat{Q}(\Gamma_k)) \leq D(\Gamma)(1 - \delta \rho_{\Gamma_k}(v_0)).$$

In other words, in case (i), making a ε-small perturbation of P if necessary, if a difference v_0 appears with a positive frequency in Γ, then some difference $v_1 \neq 0$ will also appear with positive frequency, with the fundamental property that $\|v_1\|_2 < \|v_0\|_2$. In case (ii), the rate of injectivity strictly decreases between times k and $k+1$.

We then iterate this process, as long as we are in the first case of the lemma: starting from a difference v_0 appearing with a frequency ρ_0 in Γ_k, one can build a sequence of differences (v_m) of vectors of $\mathbf{Z}^n$ with decreasing norm such that for every m we have $\rho_{\Gamma_{k+m}}(v_m) \geq \delta^m \rho_0$. Ultimately, this sequence of points (v_m) will go to 0 (as it is a sequence of integral points with decreasing norms). Thus, there will exist a rank $m_0 \leq \|v_0\|_2^2$ such that we will be in case (ii) of the lemma (which is the only case occurring when $\|v_0\|_2 = 1$). Then, we will get

$$D(\Gamma_{k+m_0}) \leq D(\Gamma_k)(1 - \delta^{m_0} \rho_\Gamma(v_0)).$$

It remains to initialize this construction, that is, to find a difference $v_0 \in \mathbf{Z}^n$ "not too far from 0" and such that $\rho_\Gamma(v_0)$ is large enough. This step simply consists in the

[5] Unfortunately, there is a misprint in the first inequality of (ii): at the left side a constant depending only on $\|P^{-1}\|$ is missing.

application of Theorem 3.2: applying it to $S = B(0, r)$ with $\mu_n (r/2)^n = 2/D(\Gamma)$, one gets

$$\sum_{u \in B(0,r)} \rho_\Gamma(u) \geq 2,$$

thus

$$\sum_{u \in B(0,r)\setminus\{0\}} \rho_\Gamma(u) \geq 1.$$

As the support of ρ_Γ is included in $\mathbf{Z}^n$, and as $\#(B(0,r) \cap \mathbf{Z}^n) \leq \mu_n (r+1)^n$, this implies that there exists $u_0 \in B(0,r) \cap (\mathbf{Z}^n \setminus \{0\})$ such that

$$\rho_\Gamma(u_0) \geq \frac{1}{\mu_n (r+1)^n},$$

which gives for $r \geq 1$

$$\rho_\Gamma(u_0) \geq \frac{D(\Gamma)}{2^{2n+1}}.$$

This allows to estimate the "loss of injectivity" $D(\Gamma_k) - D(\Gamma_{k+m_0})$ that occurs between times k and $k + m_0$. Theorem 5.4 is obtained by applying this reasoning many times.

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